13 DERIVATIVES

DEFINITION OF A DERIVATIVE

If $y = f(x)$, the derivative of y or $f(x)$ with respect to x is defined as

13.1 $$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

where $h = \Delta x$. The derivative is also denoted by y', df/dx or $f'(x)$. The process of taking a derivative is called *differentiation.*

GENERAL RULES OF DIFFERENTIATION

In the following, u, v, w are functions of x; a, b, c, n are constants [restricted if indicated]; $e = 2.71828\ldots$ is the natural base of logarithms; $\ln u$ is the natural logarithm of u [i.e. the logarithm to the base e] where it is assumed that $u > 0$ and all angles are in radians.

13.2 $$\frac{d}{dx}(c) = 0$$

13.3 $$\frac{d}{dx}(cx) = c$$

13.4 $$\frac{d}{dx}(cx^n) = ncx^{n-1}$$

13.5 $$\frac{d}{dx}(u \pm v \pm w \pm \cdots) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \cdots$$

13.6 $$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

13.7 $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

13.8 $$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

13.9 $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$$

13.10 $$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

13.11 $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{(Chain rule)}$$

13.12 $$\frac{du}{dx} = \frac{1}{dx/du}$$

13.13 $$\frac{dy}{dx} = \frac{dy/du}{dx/du}$$

DERIVATIVES OF TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

13.14 $\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$

13.15 $\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$

13.16 $\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$

13.17 $\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}$

13.18 $\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx}$

13.19 $\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}$

13.20 $\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \left[-\frac{\pi}{2} < \sin^{-1} u < \frac{\pi}{2}\right]$

13.21 $\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad [0 < \cos^{-1} u < \pi]$

13.22 $\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx} \qquad \left[-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2}\right]$

13.23 $\frac{d}{dx}\cot^{-1} u = \frac{-1}{1+u^2}\frac{du}{dx} \qquad [0 < \cot^{-1} u < \pi]$

13.24 $\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} = \frac{\pm 1}{u\sqrt{u^2-1}}\frac{du}{dx} \qquad \begin{bmatrix} + \text{ if } 0 < \sec^{-1} u < \pi/2 \\ - \text{ if } \pi/2 < \sec^{-1} u < \pi \end{bmatrix}$

13.25 $\frac{d}{dx}\csc^{-1} u = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx} = \frac{\mp 1}{u\sqrt{u^2-1}}\frac{du}{dx} \qquad \begin{bmatrix} - \text{ if } 0 < \csc^{-1} u < \pi/2 \\ + \text{ if } -\pi/2 < \csc^{-1} u < 0 \end{bmatrix}$

DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

13.26 $\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx} \qquad a \neq 0, 1$

13.27 $\frac{d}{dx}\ln u = \frac{d}{dx}\log_e u = \frac{1}{u}\frac{du}{dx}$

13.28 $\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$

13.29 $\frac{d}{dx}e^u = e^u \frac{du}{dx}$

13.30 $\frac{d}{dx}u^v = \frac{d}{dx}e^{v \ln u} = e^{v \ln u}\frac{d}{dx}[v \ln u] = vu^{v-1}\frac{du}{dx} + u^v \ln u \frac{dv}{dx}$

DERIVATIVES OF HYPERBOLIC AND INVERSE HYPERBOLIC FUNCTIONS

13.31 $\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}$

13.32 $\frac{d}{dx}\cosh u = \sinh u \frac{du}{dx}$

13.33 $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}$

13.34 $\frac{d}{dx}\coth u = -\operatorname{csch}^2 u \frac{du}{dx}$

13.35 $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$

13.36 $\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \frac{du}{dx}$

13.37 $$\frac{d}{dx}\sinh^{-1}u \;=\; \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$$

13.38 $$\frac{d}{dx}\cosh^{-1}u \;=\; \frac{\pm 1}{\sqrt{u^2-1}}\frac{du}{dx} \qquad \begin{bmatrix} + \text{ if } \cosh^{-1}u > 0,\ u > 1 \\ - \text{ if } \cosh^{-1}u < 0,\ u > 1 \end{bmatrix}$$

13.39 $$\frac{d}{dx}\tanh^{-1}u \;=\; \frac{1}{1-u^2}\frac{du}{dx} \qquad [-1 < u < 1]$$

13.40 $$\frac{d}{dx}\coth^{-1}u \;=\; \frac{1}{1-u^2}\frac{du}{dx} \qquad [u > 1 \text{ or } u < -1]$$

13.41 $$\frac{d}{dx}\operatorname{sech}^{-1}u \;=\; \frac{\mp 1}{u\sqrt{1-u^2}}\frac{du}{dx} \qquad \begin{bmatrix} - \text{ if } \operatorname{sech}^{-1}u > 0,\ 0 < u < 1 \\ + \text{ if } \operatorname{sech}^{-1}u < 0,\ 0 < u < 1 \end{bmatrix}$$

13.42 $$\frac{d}{dx}\operatorname{csch}^{-1}u \;=\; \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx} \;=\; \frac{\mp 1}{u\sqrt{1+u^2}}\frac{du}{dx} \qquad [- \text{ if } u > 0,\ + \text{ if } u < 0]$$

HIGHER DERIVATIVES

The second, third and higher derivatives are defined as follows.

13.43 $$\text{Second derivative} \;=\; \frac{d}{dx}\left(\frac{dy}{dx}\right) \;=\; \frac{d^2y}{dx^2} \;=\; f''(x) \;=\; y''$$

13.44 $$\text{Third derivative} \;=\; \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) \;=\; \frac{d^3y}{dx^3} \;=\; f'''(x) \;=\; y'''$$

13.45 $$n\text{th derivative} \;=\; \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \;=\; \frac{d^ny}{dx^n} \;=\; f^{(n)}(x) \;=\; y^{(n)}$$

LEIBNITZ'S RULE FOR HIGHER DERIVATIVES OF PRODUCTS

Let D^p stand for the operator $\frac{d^p}{dx^p}$ so that $D^pu = \frac{d^pu}{dx^p}$ = the pth derivative of u. Then

13.46 $$D^n(uv) \;=\; uD^nv + \binom{n}{1}(Du)(D^{n-1}v) + \binom{n}{2}(D^2u)(D^{n-2}v) + \cdots + vD^nu$$

where $\binom{n}{1}, \binom{n}{2}, \ldots$ are the binomial coefficients [page 3].

As special cases we have

13.47 $$\frac{d^2}{dx^2}(uv) \;=\; u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

13.48 $$\frac{d^3}{dx^3}(uv) \;=\; u\frac{d^3v}{dx^3} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + v\frac{d^3u}{dx^3}$$

DIFFERENTIALS

Let $y = f(x)$ and $\Delta y = f(x+\Delta x) - f(x)$. Then

13.49 $$\frac{\Delta y}{\Delta x} \;=\; \frac{f(x+\Delta x)-f(x)}{\Delta x} \;=\; f'(x) + \epsilon \;=\; \frac{dy}{dx} + \epsilon$$

where $\epsilon \to 0$ as $\Delta x \to 0$. Thus

13.50 $$\Delta y \;=\; f'(x)\,\Delta x + \epsilon\,\Delta x$$

If we call $\Delta x = dx$ the differential of x, then we define the differential of y to be

13.51 $$dy \;=\; f'(x)\,dx$$

RULES FOR DIFFERENTIALS

The rules for differentials are exactly analogous to those for derivatives. As examples we observe that

13.52 $$d(u \pm v \pm w \pm \cdots) = du \pm dv \pm dw \pm \cdots$$

13.53 $$d(uv) = u\,dv + v\,du$$

13.54 $$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

13.55 $$d(u^n) = nu^{n-1}\,du$$

13.56 $$d(\sin u) = \cos u\,du$$

13.57 $$d(\cos u) = -\sin u\,du$$

PARTIAL DERIVATIVES

Let $f(x, y)$ be a function of the two variables x and y. Then we define the partial derivative of $f(x, y)$ with respect to x, keeping y constant, to be

13.58 $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Similarly the partial derivative of $f(x, y)$ with respect to y, keeping x constant, is defined to be

13.59 $$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Partial derivatives of higher order can be defined as follows.

13.60 $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

13.61 $$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

The results in 13.61 will be equal if the function and its partial derivatives are continuous, i.e. in such case the order of differentiation makes no difference.

The differential of $f(x, y)$ is defined as

13.62 $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

where $dx = \Delta x$ and $dy = \Delta y$.

Extension to functions of more than two variables are exactly analogous.

14 INDEFINITE INTEGRALS

DEFINITION OF AN INDEFINITE INTEGRAL

If $\dfrac{dy}{dx} = f(x)$, then y is the function whose derivative is $f(x)$ and is called the *anti-derivative* of $f(x)$ or the *indefinite integral* of $f(x)$, denoted by $\int f(x)\,dx$. Similarly if $y = \int f(u)\,du$, then $\dfrac{dy}{du} = f(u)$. Since the derivative of a constant is zero, all indefinite integrals differ by an arbitrary constant.

For the definition of a definite integral, see page 94. The process of finding an integral is called *integration.*

GENERAL RULES OF INTEGRATION

In the following, u, v, w are functions of x; a, b, p, q, n any constants, restricted if indicated; $e = 2.71828\ldots$ is the natural base of logarithms; $\ln u$ denotes the natural logarithm of u where it is assumed that $u > 0$ [in general, to extend formulas to cases where $u < 0$ as well, replace $\ln u$ by $\ln |u|$]; all angles are in radians; all constants of integration are omitted but implied.

14.1 $$\int a\,dx = ax$$

14.2 $$\int af(x)\,dx = a\int f(x)\,dx$$

14.3 $$\int (u \pm v \pm w \pm \cdots)\,dx = \int u\,dx \pm \int v\,dx \pm \int w\,dx \pm \cdots$$

14.4 $$\int u\,dv = uv - \int v\,du \qquad \text{[Integration by parts]}$$

For generalized integration by parts, see 14.48.

14.5 $$\int f(ax)\,dx = \frac{1}{a}\int f(u)\,du$$

14.6 $$\int F\{f(x)\}\,dx = \int F(u)\frac{dx}{du}\,du = \int \frac{F(u)}{f'(x)}\,du \quad \text{where } u = f(x)$$

14.7 $$\int u^n\,du = \frac{u^{n+1}}{n+1}, \quad n \neq -1 \qquad \text{[For } n = -1\text{, see 14.8]}$$

14.8 $$\int \frac{du}{u} = \ln u \quad \text{if } u > 0 \text{ or } \ln(-u) \text{ if } u < 0$$
$$= \ln |u|$$

14.9 $$\int e^u\,du = e^u$$

14.10 $$\int a^u\,du = \int e^{u \ln a}\,du = \frac{e^{u \ln a}}{\ln a} = \frac{a^u}{\ln a}, \quad a > 0,\ a \neq 1$$

14.11 $\int \sin u\, du = -\cos u$

14.12 $\int \cos u\, du = \sin u$

14.13 $\int \tan u\, du = \ln \sec u = -\ln \cos u$

14.14 $\int \cot u\, du = \ln \sin u$

14.15 $\int \sec u\, du = \ln(\sec u + \tan u) = \ln \tan\left(\frac{u}{2}+\frac{\pi}{4}\right)$

14.16 $\int \csc u\, du = \ln(\csc u - \cot u) = \ln \tan\frac{u}{2}$

14.17 $\int \sec^2 u\, du = \tan u$

14.18 $\int \csc^2 u\, du = -\cot u$

14.19 $\int \tan^2 u\, du = \tan u - u$

14.20 $\int \cot^2 u\, du = -\cot u - u$

14.21 $\int \sin^2 u\, du = \frac{u}{2} - \frac{\sin 2u}{4} = \frac{1}{2}(u - \sin u \cos u)$

14.22 $\int \cos^2 u\, du = \frac{u}{2} + \frac{\sin 2u}{4} = \frac{1}{2}(u + \sin u \cos u)$

14.23 $\int \sec u \tan u\, du = \sec u$

14.24 $\int \csc u \cot u\, du = -\csc u$

14.25 $\int \sinh u\, du = \cosh u$

14.26 $\int \cosh u\, du = \sinh u$

14.27 $\int \tanh u\, du = \ln \cosh u$

14.28 $\int \coth u\, du = \ln \sinh u$

14.29 $\int \operatorname{sech} u\, du = \sin^{-1}(\tanh u)$ or $2\tan^{-1} e^u$

14.30 $\int \operatorname{csch} u\, du = \ln \tanh\frac{u}{2}$ or $-\coth^{-1} e^u$

14.31 $\int \operatorname{sech}^2 u\, du = \tanh u$

14.32 $\int \operatorname{csch}^2 u\, du = -\coth u$

14.33 $\int \tanh^2 u\, du = u - \tanh u$